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Linear Stability of Quasi-two-dimensional Liquid Metal Duct Flow Subjected to a Transverse Magnetic Field and Vertical Temperature Stratification

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Abstract

The linear stability of a magnetohydrodynamic duct flow with heating from below is investigated and serves as an extension to the class of Poiseuille–Rayleigh–Bénard flows. Such flows can be found in the blankets of nuclear fusion reactors and have been of great interest in recent times due to the endeavour of demonstrating the viability of nuclear fusion as a future energy source. The flow is described by the quasi-two-dimensional model proposed by Sommeria and Moreau [12] coupled with the energy transport equation through the Boussinesq approximation. The onset of several instability modes in this system is studied as functions of the non-dimensional governing parameters: Reynolds number Re, Rayleigh number Ra, and modified Hartmann number H. As $H \to 0$, the classical plane plane Poiseuille and Rayleigh–Bénard flows are recovered with critical values that are in agreement with previous literature. As $H \to \infty$, the relationships $Re_c \propto H^{1/2}$ and $Ra_c \propto H$ are obtained. Neutral stability curves are obtained for fixed values of low Re and low Ra conditions and are mapped onto Ra_c–H and Re_c–H regimes, respectively. The critical eigenmodes feature characteristics that are consistent with Tollmien–Schlichting waves and Rayleigh–Bénard convection cells.

Introduction

The linear stability of an electrically conducting fluid flowing through a rectangular duct subjected to a transverse magnetic field and heating from below is investigated. There has been growing interest in magnetohydrodynamic flows and fusion-related applications in recent years due to its potentially significant contribution as a future energy source. The realisation of this potential energy source is most notably championed through the International Thermonuclear Experimental Reactor (ITER) project which is funded by several international entities.

This fundamental study is motivated in part by the desire to enhance heat transfer in the blanket ducts of nuclear fusion reactors. The primary issue lies within the strong transverse magnetic field used to confine the plasma, as it causes the flow to become quasi-two-dimensional and restricts the growth of disturbances to the horizontal walls. This consequence is detrimental to the heat transfer properties of the flow. Several studies have demonstrated heat transfer improvement with the implementation of turbulent promoters such as bluff bodies [2] and imposed external forcing such as current injection [5]. However, these flow modifiers are not always practicable and therefore the characterisation of flow instabilities is required.

The stability of a magnetohydrodynamic duct flow to quasi-two-dimensional perturbations under a transverse magnetic field, without consideration of thermal stratification, has been investigated previously by Pothérat [9]. It was found that the critical modes of linear instability are Tollmien–Schlichting waves. An asymptotic regime was observed for $H \gtrsim 2000$ (large ratio between electromagnetic and viscous forces), with the critical Reynolds number and streamwise wavenumber asymptotic relationships described by $Re_c = 4.83504 \times 10^4 H^{1/2}$ and $k_c = 0.161532 H^{1/2}$, respectively. The results indicate that the stability of the system in the limit of $H \to \infty$ is determined only by the thickness of the Shercliff layer, which scales according to $\delta_S \propto 1/H^{1/2}$.

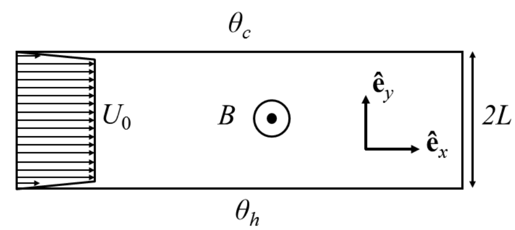

Figure 1: The system under investigation with duct height $2L$, magnetic field B and the peak velocity U_0. Unit vectors $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y)$ point in the (x, y) directions, respectively. The liquid metal flows in the positive-x direction while the magnetic field acts in the z direction (out-of-plane). Temperatures θ_h (hot) and θ_c (cold) are imposed on the bottom and top duct walls, respectively.

This paper extends the work of [9] through a linear stability analysis of the same system with added thermal stratification.

Methodology

Problem Description

The system studied represents an electrically conducting fluid with kinematic viscosity ν, thermal diffusivity κ, volumetric expansion coefficient α, density ρ, and electrical conductivity σ, flowing through a horizontal rectangular duct of height $2L$ and width a, exposed to a transverse magnetic field of strength B and a vertical temperature gradient. A schematic of the system is shown in figure 1. The duct walls are electrically insulated. Provided that the imposed transverse magnetic field is sufficiently strong relative to the through-flow, the flow solution can be described accurately by the quasi-two-dimensional model proposed by Sommeria and Moreau [12] (referred to as the SM82 model hereafter). This is a result of the magnetic field suppressing motions and gradients parallel to the magnetic field in the interior of the flow. The SM82 model is used widely in the field of magnetohydrodynamics as it reduces the dimension of the problem to two-dimensions, and eliminates the need to resolve the Hartmann layers, which are the smallest-scale structures in the system. Furthermore, the model has demonstrated results consistent with three-dimensional magnetohydrodynamic flows. Derivation and details of the SM82 model can be found in [12, 10].

Governing Equations and Parameters

The SM82 equations are coupled with a thermal transport equation through a Boussinesq approximation to describe the magnetohydrodynamic duct flow with vertical thermal stratification. The non-dimensional equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u} - \frac{H}{Re}\mathbf{u} + \frac{Ra}{PrRe^2}\theta\hat{\mathbf{e}}_y, \quad (1a)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = \frac{1}{PrRe}\nabla^2\theta \quad (1b)$$

$$\nabla\cdot\mathbf{u} = 0, \quad (1c)$$

are obtained by normalising lengths by the half the duct height L, velocity by the maximum velocity of the base flow profile U_0, time by L/U_0, pressure by ρU_0^2 and temperature by $\Delta\theta = \theta_h - \theta_c$ (temperature difference between the hot and cold walls). Here, $\hat{\mathbf{e}}_y$ is a unit vector in positive y direction. The non-dimensional parameters Re, Ra, H, and Pr are respectively defined as

$$Re = \frac{U_0 L}{\nu},\ \ Ra = \frac{\alpha g L^3 \Delta\theta}{\nu\kappa},\ \ H = nHa\left(\frac{L}{a}\right)^2,\ \ Pr = \frac{\nu}{\kappa}, \quad (2)$$

where $n = 2$ the number of Hartmann layers on out-of-plane walls imparting friction on the quasi-two-dimensional flow, and Ha is the Hartmann number describing the relative influence of magnetic to viscous forces on the flow, defined by $Ha = aB\sqrt{\sigma/(\rho\nu)}$. The Prandtl number (ratio of momentum diffusivity to thermal diffusivity) is fixed at $Pr = 0.022$ to represent the eutectic liquid metal alloy Galinstan (GaInSn) that is used in a number of modern magnetohydrodynamic experiments [8]. Equation 1 is used for flows with both through-flow and heating (i.e. $Re > 0$, $Ra > 0$). Different normalisations are adopted to facilitate computations of the bookend cases $Re = 0$ and $Ra = 0$. The details have been omitted here.

The base flow solutions for velocity $\overline{\mathbf{u}} = \bar{u}(y)\hat{\mathbf{e}}_x$ and temperature $\bar{\theta}(y)\hat{\mathbf{e}}_x$ are taken to be fully developed (i.e. time invariant and dependent only on y), and are therefore defined based on the known analytical expressions. The base temperature profile is given by $\bar{\theta}(y) = (1-y)/2$. For $H = 0$, the base velocity profile is described by $\bar{u}(y) = 1 - y^2$ and for $H > 0$,

$$\bar{u}(y) = \left(\frac{\cosh(\sqrt{H})}{\cosh(\sqrt{H}) - 1}\right)\left(1 - \frac{\cosh(\sqrt{H}y)}{\cosh(\sqrt{H})}\right). \quad (3)$$

Linear Stability Analysis

The governing equations are linearised by decomposing velocity, temperature and pressure solutions into the mean component (base flow) and a fluctuating component (perturbation). That is, $f = \bar{f} + f'$ where the overbar and prime symbols represent the mean and fluctuating quantities, respectively, and f is any of u, v, p or θ. Due to the translational invariance of the problem in the x direction, the perturbations comprise travelling waves in the x direction and take the modal form of $f' = \delta\tilde{f}(y)\exp[\mathrm{i}(kx - \omega t)]$, where δ is taken to be a small parameter, k is the streamwise wavenumber, ω is a complex eigenvalue related to the frequency and growth rate of the eigenvector, and the tilde components $(\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\theta})$ are eigenfunctions. Substituting these expressions into equation (1) (for $Re > 0$, $Ra > 0$) and retaining the terms up to order $\mathscr{O}(\delta)$ yields

$$\begin{aligned}&\frac{1}{Re}\mathbb{M}^2\tilde{v} + \mathrm{i}k\bar{u}''\tilde{v} - \mathrm{i}k\bar{u}\mathbb{M}\tilde{v} \\ &\quad - \frac{H}{Re}\mathbb{M}\tilde{v} - \frac{Ra}{Re^2 Pr}k^2\tilde{\theta} = -\mathrm{i}\omega\mathbb{M}\tilde{v},\end{aligned} \quad (4a)$$

$$-\bar{\theta}'\tilde{v} - \mathrm{i}k\bar{u}\tilde{\theta} + \frac{1}{RePr}\mathbb{M}\tilde{\theta} = -\mathrm{i}\omega\tilde{\theta}, \quad (4b)$$

where D is the differentiation operator with respect to y and $\mathbb{M} = \mathrm{D}^2 - k^2$. The linearised equations are limited to transverse rolls since longitudinal rolls are outside the scope of the SM82 model and therefore not considered in this study.

The linearised governing equations are treated as an eigenvalue problem posed in the standard form following McBain, Chubb and Armfield [7], whereby the eigenvectors contain $\tilde{v}$ and $\tilde{\theta}$. A MATLAB eigenvalue solver is used to obtain the leading eigenvalues and corresponding eigenvectors for equation (4) ($Re > 0$, $Ra > 0$). To begin searching for the critical conditions, two of the three governing parameters (either Re, Ra or H) are fixed while the third parameter is varied to seek $\mathrm{Im}\{\omega\} = 0$. The flow condition is considered to be critical once the varying parameter (either Re_c, Ra_c or H_c) and corresponding growth rate have converged to at least 5 significant figures.

The recovered critical values for Rayleigh–Bénard and Poiseuille flow were recovered and agree very well with previous literature [3, 4]. Additional validation studies were conducted but are not presented here. The present code has been successfully implemented for flows driven by horizontal convection [13] and is based on several numerical methodologies referenced therein.

Results and Discussion

Stability for $Re = 0$ and $Ra = 0$

The neutral stability curves for Re_c for a range of Rayleigh numbers ($0 \le Ra \le 1\times 10^3$) and the corresponding critical wavenumbers k_c are provided in figure 2(a). This section is first dedicated to $Ra = 0$. An excellent agreement in Re_c values between the present study and [9] was achieved with a maximum percentage error of less than 0.1% across the range of $10^{-2} \le H \le 10^4$. The neutral stability curve for $Ra = 0$ demonstrates that Re_c increases monotonically with increasing H which is due to a stronger through-flow being required to counteract the increased damping induced by the increased magnetic field strength. As $H \to 0$, the flow becomes purely hydrodynamic and is unstable to Tollmien–Schlichting waves (shear-dominant instability) at $Re_c = 5772.22$ [4]. As $H \to \infty$, the onset of instability is described by an asymptotic trend regime $Re_c = 48347H^{1/2}$ for $H \gtrsim 2000$ governed by the Shercliff layers. The streamwise wavenumber of the instability is a constant $k_c = 1.02$ at low H and adopts an asymptotic trend at high H described by $k_c = 0.1615H^{1/2}$.

The neutral stability curves for Ra_c at various Reynolds numbers ($0 \le Re \le 300$) and the corresponding wavenumbers are shown in figure 2(b). This section is focused on $Re = 0$ (represented by the dashed-line in the figure). As $H \to 0$, classical plane Rayleigh–Bénard flow is recovered and the perturbations are controlled by the balance between buoyancy and viscous dissipation. Therefore, the appearance of the convection cells takes place at constant $Ra_c = 213.47$ (corresponding to $Ra = 1707.7$ based on full duct-height length scale). However, as $H \to \infty$ the neutral stability curve approaches a trend described by $Ra_c = 21.672H^{0.991}$ for $2000 \le H \le 10^4$. It is expected here that the exponent 0.991 will approach unity as $H \to \infty$ and may be understood as follows. In the limit $H \to \infty$, Hartmann friction dominates and has to be balanced by buoyancy for convection to set in. Hence, at the onset the balance between these forces implies $Ra \sim H$, which scales with the duct height $2L$. The critical wavenumber of the instability remains relatively constant across all H, with $k_c \to 1.5582$ as $H \to 0$, and $k_c \to \pi/2$ as $H \to \infty$. The Ra_c and k_c scalings are in agreement with the linear stability analysis performed by Burr and Müller [1] who studied Rayleigh–Bénard convection in liquid metal

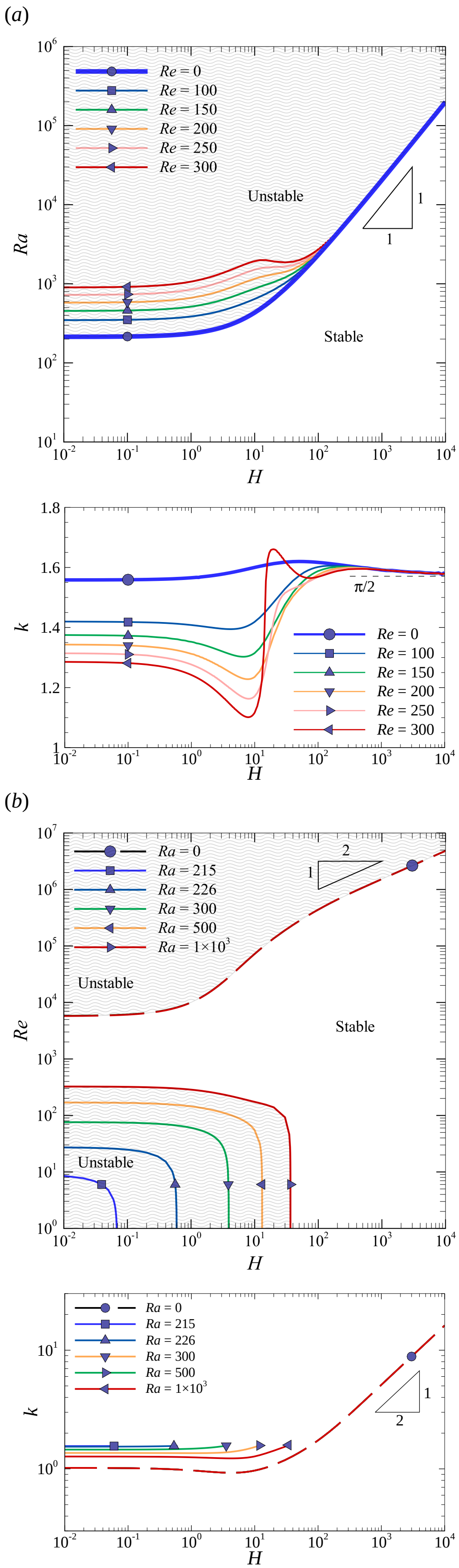

Figure 2: Neutral stability curves of (a) Ra_c and (b) Re_c and their corresponding k_c. The symbols identify the Re or Ra for each curve. Shaded regions represent unstable flow conditions.

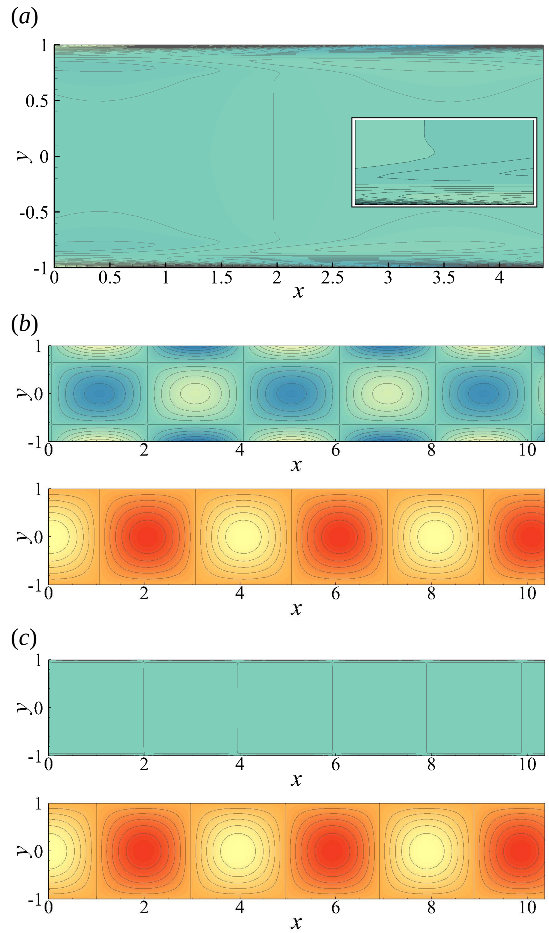

Figure 3: (a) Vorticity in the eigenvector field of $(Re, Ra, H) = (10033, 0, 1)$ for $k_c = 0.9713$ with closeup of boundary layer in the inset. Vorticity (top panel) and temperature (bottom panel) for (b) $(0, 235, 1)$ with $k_c = 1.5657$, and (c) $(0, 199513, 1\times10^4)$ with $k_c = 1.5929$. Dark and light shading represent negative and positive values, respectively.

layers under the influence of a horizontal magnetic field. The streamwise wavenumber scaling for both $Re = 0$ and $Ra = 0$ cases as $H \to \infty$ support the view that the instability scales with the Shercliff layer thickness and channel height, respectively.

A typical plot of vorticity in the eigenvector field for Re_c with $Ra = 0$ (no heating) and $H = 1$ is illustrated in figure 3(a). The instability caused by the shearing through-flow is most dominant along the horizontal walls in the boundary layers. Further increasing H only results in thinner Shercliff layers and therefore decreasing the region occupied by instability disturbance structures (similar to figure 3c). For all H, the leading eigenvalue was consistently found to lie on the A branch of the eigenvalue spectrum (referred to as wall modes [11]), in agreement with [9]. For the case of heating and no through-flow, it is found that the vorticity disturbances occupy both regions along the walls and throughout the interior of the duct as shown in figure 3(b)(top panel) for $H = 1$. The interior disturbances are evident only at low H as the strength of the counter-rotating Rayleigh–Bénard-like cells are comparable to the wall disturbances. Increases to the magnetic field strength causes the vorticity in the interior to weaken and diminish and ultimately lead to an eigenvector field that is devoid of any vorticity disturbances in the interior. Despite this, the structure of the thermal disturbances appear to be unaffected and is insensitive to H.

Stability for $0 \leq Re \leq 300$

Figure 2(*a*) shows neutral stability curves for several Reynolds numbers over $0 \leq Re \leq 300$. Throughout this range of *Re*, the neutral stability curves maintain their continuous profiles with slight changes with increasing *Re*. These curves reveal that Ra_c is Reynolds number-dependent in the limit $H \to 0$, but develops an independent relationship described by $Ra_c = 21.672H^{0.991}$ as $H \to \infty$. As *H* increases, the Hartmann friction becomes the dominant damping process over viscous dissipation acting on the instabilities and therefore at sufficiently large *H*, the instability threshold becomes insensitive to the Reynolds number. However, for low to moderate values of *H*, the viscous dissipation influences Ra_c causing it to increase with increasing *Re*. The stabilisation can be explained by the disruption in forming recirculating convection cells caused by the shear and therefore a stronger thermal gradient is required to overcome the through-flow. This is a well known result and demonstrated by experiments by Luijkx, Platten, and Legros [6]. Correspondingly, increasing *Re* demonstrates a decrease in k_c (i.e. increased wavelength) caused by the stronger shear which elongates the forming convection cells. The increase in k_c at intermediate values of *H* is related to the change in force balances at the onset of convection cells. At higher *H*, the instability is determined solely by the Hartmann layers and therefore k_c is independent of *Re*.

Stability for $0 \leq Ra \leq 1 \times 10^3$

The stability of finite *Ra* flows introduces a thermal-dominant instability in addition to the shear-dominant instability observed at $Ra = 0$. The stability of both these modes are presented in figure 2(*b*) as solid lines and dashed-lines, respectively. The shear-dominant instability is a Tollmien–Schlichting wave and is found to be weakly sensitive to all *Ra* investigated and therefore appears as a single curve. In contrast, the thermal-dominant instability is very sensitive to *Ra* and first develops for $Ra = 213.47$ provided that the through-flow is sufficiently weak. A strong through-flow suppresses the development of the thermoconvective cells. Hence, the solid-line curves represent the suppression of the thermal instability and is the reason for why instability exists below the curves as opposed to above the curves. Therefore, as $Re \to 0$ the flow becomes more prone to thermal instability whereas the flow is more susceptible to shear instability as $Re \to \infty$. This result demonstrates that increasing the Reynolds number acts to suppress the thermally-dominant transverse rolls as has been discussed previously. However, this approach also demonstrates a progression from an unstable flow to a stable flow, and to an unstable flow again with increasing *Re*, which was not observable in the *Ra-H* stability diagram.

The *Re*–*H* stability diagram shows that thermal disturbances are weakly sensitive to the magnetic damping at low *H* but are abruptly suppressed when the magnetic field strength is sufficiently strong. Indeed the *H* values corresponding to the vertical neutral curves are precisely the critical values demonstrated in the *Ra*–*H* regime for $Re = 0$ (see figure 2*a*). The critical wavenumbers associated with the shear-dominant instability (as shown in figure 2*b*) are weakly sensitive to the range of *Ra* investigated here, which is not surprising given that Re_c is also weakly sensitive. The thermal-dominant instability adopts a larger wavenumber structure which decreases with increasing *Ra*. The eigenvectors appear similar to those shown in figure 3.

Conclusions

This paper describes the linear stability of Poiseuille–Rayleigh–Bénard flows under the effect of a transverse magnetic field for low *Re* and low *Ra* conditions. The onset of shear-dominant instability and suppression of thermal-dominant instability were determined and mapped onto stability diagrams over the range of $0 \leq H \leq 10^4$. Asymptotic relationships described by $Re_c \propto H^{1/2}$ and $Ra_c \propto H$ as $H \to \infty$ were obtained. In the former case, the flow stability is governed by the individual Shercliff layers while in the latter case, the balance between buoyancy and Hartmann friction dictates stability.

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